A Generalization of the Ramanujan Polynomials and Plane Trees

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Abstract. Generalizing a sequence of Lambert, Cayley and Ramanujan, Chapoton has recently introduced a polynomial sequence $Q_n := Q_n(x, y, z, t)$ defined by

$$Q_1 = 1, \quad Q_{n+1} = [x + nz + (y + t)(n + y\partial_y)]Q_n.$$

In this paper we prove Chapoton's conjecture on the duality formula: $Q_n(x, y, z, t) = Q_n(x+nz+nt, y, -t, -z)$, and answer his question about the combinatorial interpretation of Q_n. Actually we give combinatorial interpretations of these polynomials in terms of plane trees, half-mobile trees, and forests of plane trees. Our approach also leads to a general formula that unifies several known results for enumerating trees and plane trees.

Keywords: Ramanujan polynomials, plane tree, half-mobile tree, forest, general descent, elder vertex, improper edge

MR Subject Classifications: Primary 05A15; Secondary 05C05

1 Introduction

It is well-known that the Lambert function $w = \sum_{n \geq 1} n^{n-1}y^n/n!$ (see [9]) is the solution to the functional equation $we^{-w} = y$ with $w(0) = 0$ and a formula of Cayley [2] says that n^{n-1} counts the rooted labeled trees on n vertices. There are various generalizations of Cayley's formula. In particular, an interesting refinement of the sequence n^{n-1} appeared in Ramanujan's work (see [1,6,7,13]) and is related to Lambert's series as follows: Differentiating n times Lambert's function w with respect to y (see [13, Lemma 6]) yields

$$w^{(n)} = \frac{e^{nw}}{(1-w)^n} R_n\left(\frac{1}{1-w}\right), \tag{1.1}$$

where R_n is a polynomial of degree $n-1$ and satisfies the recurrence:

$$R_1 = 1, \quad R_{n+1}(y) = [n(1+y) + y^2\partial_y]R_n(y). \tag{1.2}$$

It follows that $R_2 = 1 + y$, $R_3 = 2 + 4y + 3y^2$ and $R_4 = 6 + 18y + 25y^2 + 15y^3$. Clearly formula (1.2) implies that $R_n(y)$ is a polynomial with nonnegative integral coefficients such that $R_n(0) = (n-1)!$ and the leading coefficient is $(2n-3)!!$. As the Lambert function is equivalent to $w^{(n)}(0) = n^{n-1}$, we derive from (1.1) that $R_n(1) = n^{n-1}$.

On the other hand, Shor [10] and Dumont-Ramamonjisoa [7] have proved independently that the coefficient of y^k in $R_n(y)$ counts *rooted labeled trees* on n vertices with k "improper edges." In a subsequent paper [13, Eq. (26)] Zeng proved that if we set

$$Z = \frac{e^{xw} - 1}{x} = w + \frac{w^2}{2!}x + \frac{w^3}{3!}x^2 + \cdots,$$

then differentiating n times Z with respect to y yields

$$Z^{(n)} = \frac{e^{(x+n)w}}{(1-w)^n}P_n\left(\frac{1}{1-w}, y\right),\tag{1.3}$$

where $P_n := P_n(x, y)$ is a polynomial defined by the recurrence relation:

$$P_1 = 1, \quad P_{n+1} = [x + n + y(n + y\partial_y)]P_n.$$

The polynomials $P_n(x, y)$ are later called the *Ramanujan polynomials* [3,6].

Recently, Chapoton [5] generalized P_n to the polynomials $Q_n := Q_n(x, y, z, t)$ as follows:

$$Q_1 = 1, \quad Q_{n+1} = [x + nz + (y + t)(n + y\partial_y)]Q_n.\tag{1.4}$$

In the context of *operads*, Chapoton [4, p.5](see also [3]) conjectured that the coefficient of $x^i y^{j-i}$ in $Q_n(1, x, 0, y)$ is the *dimension* of homogeneous component of degree (i, j) of the so-called Ramanujan operads $\mathrm{Ram}(\{1, 2, \ldots, n\})$.

Clearly these are *homogeneous polynomials* in x, y, z, t of degree $n - 1$ and z is just a homogeneous parameter. For example, we have $Q_2 = x + y + z + t$ and

$$Q_3 = x^2 + 3xy + 3xz + 3xt + 3y^2 + 4yz + 5yt + 2z^2 + 4zt + 2t^2.$$

We can easily derive explicit product formulae of Q_n for some special values. Indeed, setting $t = -y$ in (1.4) yields

$$Q_n(x, y, z, -y) = \prod_{k=1}^{n-1}(x + kz),\tag{1.5}$$

while setting $y = 0$ in (1.4) leads to

$$Q_n(x, 0, z, t) = \prod_{k=1}^{n-1}(x + kz + kt).\tag{1.6}$$

Further factorization formulae can be derived from the following *duality formula*, which was conjectured by Chapoton [5].

Theorem 1.1. *For $n \geq 1$, there holds*

$$Q_n(x, y, z, t) = Q_n(x + nz + nt, y, -t, -z).\tag{1.7}$$

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Table 1: Values of $Q_{n,k}(x,t)$.

$k\backslash n$	1	2	3	4
0	1	$x+1+t$	$x^2+3x+2+(3x+4)t+2t^2$	$x^3+6x^2+11x+6+(6x^2+22x+18)t+(11x+18)t^2+6t^3$
1		1	$3x+4+5t$	$6x^2+22x+18+(26x+43)t+26t^2$
2			3	$15x+25+35t$
3				15
$\sum_k$	1	$x+2+t$	$(x+3+t)(x+3+2t)$	$(x+4+t)(x+4+2t)(x+4+3t)$

It follows from (1.5) and (1.7) that when $y = z$ the polynomials Q_n factorize completely into linear factors:

$$Q_n(x,z,z,t) = \prod_{k=1}^{n-1}(x+nz+kt). \tag{1.8}$$

In particular, we obtain $Q_n(1,1,1,1) = n!C_n$, where $C_n = \frac{1}{n+1}\binom{2n}{n}$ is the n-th *Catalan number*. This leads us to first look for a combinatorial interpretation in the set of *labeled plane trees* on $n+1$ vertices rooted at 1, of which the cardinality is $n!C_n$ (see [12, p. 220]). To this end, as $Q_n(x,y,z,t) = z^{n-1}Q_n(x/z,y/z,1,t/z)$, it is convenient to write $Q_n(x,y,1,t)$ as follows:

$$Q_n(x,y,1,t) = \sum_{k=0}^{n-1}Q_{n,k}(x,t)y^k. \tag{1.9}$$

Now identifying the coefficients of y^k in (1.4) we obtain $Q_{1,0}(x,t) = 1$ and for $n \geq 2$:

$$Q_{n,k}(x,t) = [x+n-1+t(n+k-1)]Q_{n-1,k}(x,t) + (n+k-2)Q_{n-1,k-1}(x,t), \quad (1.10)$$

where $Q_{n,k}(x,t) = 0$ if $k \geq n$ or $k < 0$. The first values of $Q_{n,k}(x,t)$ are given in Table 1.

In the next section we shall prove that the polynomials $Q_{n,k}(x,t)$ count plane trees of k improper edges with respect to two new statistics "eld" and "young." It turns out that the polynomials Q_n are the counterpart of Ramanujan's polynomials P_n (which count rooted trees) for plane trees.

Originally Chapoton asked for a combinatorial interpretation of Q_n in the model of *half-mobile trees*, we shall answer his question in Section 3 by establishing a bijection from plane trees to half-mobile trees.

In Section 4 we shall unify and generalize several classical formulae for the enumeration of trees and plane trees. For instance, in Theorem 4.3 we prove that

$$\sum_{T} t^{\mathrm{eld}(T)} \prod_{i=1}^{n} x_i^{\mathrm{young}_T(i)} = \prod_{k=0}^{n-2}(x_1+\cdots+x_n+kt),$$

where T ranges over all plane tree trees on $\{1,\ldots,n\}$ and eld and young are mentioned as before. In Section 5 we give another combinatorial interpretation of $Q_{n,k}(x,t)$ in terms of forests of plane trees, which extends a previous result of Shor [10]. In Section 6 we give a short proof of Theorem 1.1. We end this paper with some open problems.

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2   Combinatorial interpretations in plane trees

Let $\pi = a_1 \cdots a_n$ be a permutation of a totally ordered set of n elements. Recall that an element a_i is said to be a *right-to-left minimum* of π if $a_i < a_j$ for every $j > i$. For our purpose we need to introduce a *dual* statistic on permutations as follows. The integer i ($1 \leq i \leq n-1$) is called a *general descent* of π, if there exists a $j > i$ such that $a_j < a_i$. In other words, the general descents of π are positions that do not correspond to right-to-left minima. The number of general descents of π is denoted by $\mathrm{gdes}(\pi)$. For instance, if $\pi = 3\,6\,1\,4\,5\,8\,7$, then the general descents of π are $1, 2$ and 6, so $\mathrm{gdes}(\pi) = 3$. Let $\mathfrak{S}_n$ denote the set of all permutations of the set $[n] := \{1, \ldots, n\}$. It is easy to see that the following identity holds:

$$\sum_{\pi \in \mathfrak{S}_n} t^{\mathrm{gdes}(\pi)} = (1 + t)(1 + 2t) \cdots (1 + (n-1)t). \tag{2.1}$$

Throughout this paper, unless indicated otherwise, all trees are *rooted labeled trees* on a linearly ordered vertex set. Given two vertices i and j of a tree T we say that j is a *descendant* of i if the path from the root to j passes through i. In particular, each vertex is a descendant of itself. Let $\beta_T(i)$ be the smallest descendant of i. Furthermore, if j is a descendant of i and is also connected to i by an edge, then we say that j is a *child* of i and denote the corresponding edge by $e = (i, j)$, and if j' is another child of i, then we call j' a *brother* of j.

A *plane tree* (or *ordered tree*) is a rooted tree in which the children of each vertex are linearly ordered. From now on, by saying that $v_1, \ldots, v_m$ are all the children of a vertex v of a plane tree T we mean that v_i is the i-th child of v, counting from left to right. A vertex j of a plane tree T is called *elder* if j has a brother k to its right such that $\beta_T(k) < \beta_T(j)$; otherwise we say that j is *younger*. Note that the rightmost child of any vertex is always younger. For any vertex v of a plane tree T, let $\mathrm{eld}_T(v)$ be the number of elder children of v in T. Clearly, we have

$$\mathrm{eld}_T(v) = \mathrm{gdes}(\beta_T(v_1) \cdots \beta_T(v_m)),$$

where $v_1, \ldots, v_m$ are all the children of v. Let $\mathrm{eld}(T)$ be the number of elder vertices of T. Clearly, if T is a plane tree on n vertices with $n \geq 2$, then $\mathrm{eld}(T) \leq n - 2$.

Definition 2.1. *Let $e = (i, j)$ be an edge of a tree T. We say that e is a proper edge or j is a proper child of i, if j is an elder child of i or $i < \beta_T(j)$. Otherwise, we say that e is an improper edge and j is an improper child of i.*

For example, for the plane tree T in Figure 1 the set of elder vertices is $\{3, 8, 9, 11, 12, 13\}$ and the set of the improper edges is $\{(3, 14), (4, 1), (6, 5), (10, 4), (14, 2), (14, 7)\}$.

Given a vertex v in a tree T, denote by $\deg(v)$ or $\deg_T(v)$ the number of children of v, then the number of younger children of v is given by

$$\mathrm{young}(v) = \mathrm{young}_T(v) = \deg_T(v) - \mathrm{eld}_T(v).$$

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Denote by $\mathcal{P}_{n,k}$ (respectively $\mathcal{O}_{n,k}$) the set of plane trees (respectively plane trees with root 1) on $[n]$ with k improper edges. Moreover, we may impose some conditions on the sets $\mathcal{P}_{n,k}$ and $\mathcal{O}_{n,k}$ to denote the subsets of plane trees that satisfy these conditions. For example, $\mathcal{P}_{n,k}[\deg(n) = 0]$ stands for the subset of $\mathcal{P}_{n,k}$ subject to the condition $\deg(n) = 0$.
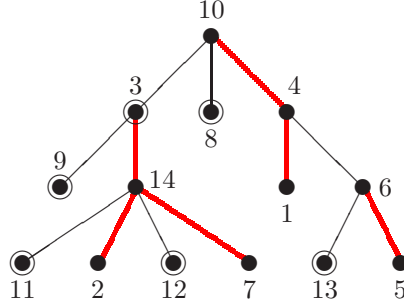

Figure 1: A plane tree on [14] with elder vertices circled and improper edges thickened.

Theorem 2.2. *The polynomials $Q_{n,k}(x,t)$ have the following interpretation:*

$$Q_{n,k}(x,t) = \sum_{T \in \mathcal{O}_{n+1,k}} x^{\mathrm{young}_T(1)-1} t^{\mathrm{eld}(T)}. \tag{2.2}$$

Proof. Clearly, identity (2.2) is true for $n = 1$. We shall prove by induction that the right-hand side of (2.2) satisfies the recurrence (1.10) by distinguishing two cases according to whether $n + 1$ is a leaf or not.

- If $T \in \mathcal{O}_{n+1,k}$ with $\deg_T(n+1) = 0$, then deleting $n+1$ yields a plane tree $T' \in \mathcal{O}_{n,k}$. Conversely, starting from any $T' \in \mathcal{O}_{n,k}$, we can recover T by adding $n+1$ to T' as a leaf in $2n - 1$ ways as follows. Pick up any vertex v of T' with the children being $a_1, \ldots, a_m$, and then add $n+1$ as the i-th $(1 \leq i \leq m+1)$ child of v to make the tree T. In other words, the children of v in T become $a_1, \ldots, a_{i-1}, n+1, a_{i+1}, \ldots, a_m$. Note that if $n + 1$ is the rightmost child of v, then $\mathrm{eld}(T) = \mathrm{eld}(T')$; otherwise, $\mathrm{eld}(T) = \mathrm{eld}(T')+1$. Meanwhile, if $n+1$ is the rightmost child of 1, then $\mathrm{young}_T(1) = \mathrm{young}_{T'}(1) + 1$; otherwise, $\mathrm{young}_T(1) = \mathrm{young}_{T'}(1)$. Since there are n vertices in T' and $\sum_{v \in [n]} \deg_{T'}(v) = n - 1$, we obtain

$$\sum_{T \in \mathcal{O}_{n+1,k}[\deg(n+1)=0]} x^{\mathrm{young}_T(1)-1} t^{\mathrm{eld}(T)}$$

$$= [x + n - 1 + t(n - 1)] \sum_{T \in \mathcal{O}_{n,k}} x^{\mathrm{young}_T(1)-1} t^{\mathrm{eld}(T)}. \tag{2.3}$$

- If $T \in \mathcal{O}_{n+1,k}$ with $\deg(n + 1) > 0$, suppose all the children of $n + 1$ are $a_1, \ldots, a_m$. Note that the edge $(n+1, a_m)$ is always younger and improper. We need to consider two cases:

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If $m = 1$ or $\beta_T(a_{m-1}) < \beta_T(a_m)$, then replace $n + 1$ by the child a_m and contract the edge joining $n + 1$ and a_m such that the original children of a_m are as the rightmost children with their previous order unchanged. Thus, we obtain a plane tree $T' \in \mathcal{O}_{n,k-1}$. Conversely, for such a T', we can recover T as follows. Pick any vertex $v \neq 1$ of T', and replace v by $n + 1$ and join v to $n + 1$ by an edge. Suppose all the children of v in T' are $b_1, \ldots, b_p$, with $b_{i_1}, b_{i_2}, \ldots, b_{i_r}$ $(i_1 < i_2 < \cdots < i_r)$ being the improper children. Then $\beta_{T'}(b_{i_j}) < \beta_{T'}(b_s)$ for any $1 \leq j \leq r$ and $s > i_j$. It is easy to see that the children of $n + 1$ in T must be $b_1, b_2, \ldots, b_{i_j}$ and v, while the children of v in T must be $b_{i_j+1}, b_{i_j+2}, \ldots, b_p$, where $0 \leq j \leq r$ and $i_0 = 0$. This means that there are $r + 1$ possibilities to partition the children of v. Since there are $k - 1$ improper edges and n vertices (one of which is 1) in T', we see that there are total $n + k - 2$ such corresponding plane trees T. Moreover, we have $\text{young}_T(1) = \text{young}_{T'}(1)$ and $\text{eld}(T) = \text{eld}(T')$. Hence the generating function for the plane trees in $\mathcal{O}_{n+1,k}$ with $n+1$ having only one child or its second child counting from right being younger is

$$(n + k - 2) \sum_{T \in \mathcal{O}_{n,k-1}} x^{\text{young}_T(1)-1} t^{\text{eld}(T)}. \tag{2.4}$$

If $m \geq 2$ and $\beta_T(a_{m-1}) > \beta_T(a_m)$, then a_{m-1} is an elder child of $n + 1$ and hence $(n + 1, a_{m-1})$ is a proper edge. Replace $n + 1$ by the child a_{m-1} and contract the edge joining $n + 1$ and a_{m-1}, such that the original children of a_{m-1} are as the rightmost children in their previous order. Thus, we obtain a plane tree $T' \in \mathcal{O}_{n,k}$. Conversely, for such a T', we can recover T similarly as the first case. Pick any vertex $v \neq 1$ of T', and replace v by $n + 1$ and join v to $n + 1$ by an edge. Suppose the children of v in T' are $b_1, \ldots, b_p$. Assume all the improper children of v in T' are $b_{i_1}, b_{i_2}, \ldots, b_{i_r}$. It is easy to see that the only possible children of $n + 1$ in T are $b_1, b_2, \ldots b_{i_j-1}, v, b_{i_j}$, while the children of v in T are $b_{i_j+1}, b_{i_j+2}, \ldots, b_p$, where $1 \leq j \leq r$. Namely there are r possibilities to partition the children of v. Since T' has k improper edges, we see that there are total k such preimages T. Moreover, we have $\text{young}_T(1) = \text{young}_{T'}(1)$ and $\text{eld}(T) = \text{eld}(T') + 1$. Hence the generating function for the plane trees in $\mathcal{O}_{n+1,k}$ with $n + 1$ having at least two children and its second child counting from right being elder is

$$k\,t \sum_{T \in \mathcal{O}_{n,k}} x^{\text{young}_T(1)-1} t^{\text{eld}(T)}. \tag{2.5}$$

Summing (2.4) and (2.5) we obtain

$$\sum_{T \in \mathcal{O}_{n+1,k}[\deg(n+1)>0]} x^{\text{young}_T(1)-1} t^{\text{eld}(T)} = k\,t \sum_{T \in \mathcal{O}_{n,k}} x^{\text{young}_T(1)-1} t^{\text{eld}(T)}$$
$$+ (n + k - 2) \sum_{T \in \mathcal{O}_{n,k-1}} x^{\text{young}_T(1)-1} t^{\text{eld}(T)}. \tag{2.6}$$

Table 2: Values of $Q_{n,k}(x-t-1,t)$.

$k\backslash n$	1	2	3	4
0	1	x	$x^2 + x + xt$	$x^3 + 3x^2 + 2x + (3x^2 + 4x)t + 2xt^2$
1		1	$3x + 1 + 2t$	$6x^2 + 10x + 2 + (14x + 7)t + 6t^2$
2			3	$15x + 10 + 20t$
3				15
$\sum_k$	1	$x+1$	$(x+2)(x+2+t)$	$(x+3)(x+3+t)(x+3+2t)$

The proof then follows from summarizing identities (2.3) and (2.6). ∎

The plane trees in $\mathcal{O}_{4,1}$ with their weights are listed in Figure 2.
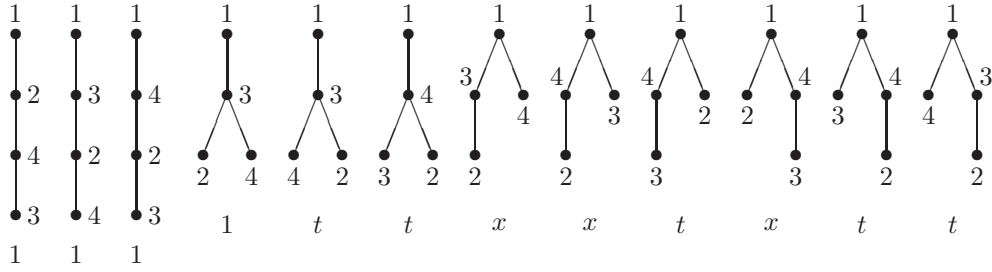

Figure 2: The polynomial $Q_{3,1}(x,t) = 3x + 4 + 5t$ as a weight function of $\mathcal{O}_{4,1}$.

Theorem 2.3. *The polynomials $Q_{n,k}(x-t-1,t)$ have the following interpretation:*

$$Q_{n,k}(x-t-1,t) = \sum_{T \in \mathcal{P}_{n,k}} x^{\mathrm{young}_T(1)} t^{\mathrm{eld}(T)}. \tag{2.7}$$

Proof. By (1.10), we see that

$$Q_{n,k}(x-t-1,t) = [x+n-2+t(n+k-2)]Q_{n-1,k}(x,t) + (n+k-2)Q_{n-1,k-1}(x,t). \tag{2.8}$$

Similarly to the proof of Theorem 2.2, we show that the right-hand side of (2.7) satisfies the recurrence (2.8). More precisely, we can prove that for $n \geq 1$,

$$\sum_{T \in \mathcal{P}_{n,k}[\deg(n)=0]} x^{\mathrm{young}_T(1)} t^{\mathrm{eld}(T)} = [x+n-2+t(n-2)] \sum_{T \in \mathcal{P}_{n-1,k}} x^{\mathrm{young}_T(1)} t^{\mathrm{eld}(T)}, \tag{2.9}$$

$$\sum_{T \in \mathcal{P}_{n,k}[\deg(n)>0]} x^{\mathrm{young}_T(1)} t^{\mathrm{eld}(T)} = k\,t \sum_{T \in \mathcal{P}_{n-1,k}} x^{\mathrm{young}_T(1)} t^{\mathrm{eld}(T)}$$

$$+ (n+k-2) \sum_{T \in \mathcal{P}_{n-1,k-1}} x^{\mathrm{young}_T(1)} t^{\mathrm{eld}(T)}. \tag{2.10}$$

The proof of (2.9) and (2.10) is exactly the same as that of (2.3) and (2.6) and is omitted here. We only mention that "Pick any vertex $v \neq 1$ of T'" needs to be changed to "Pick
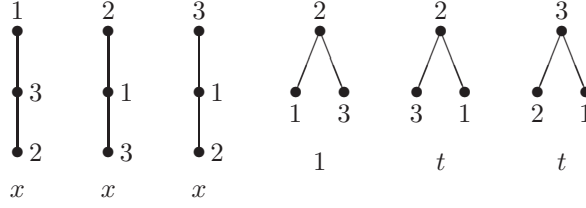
Figure 3: The polynomial $Q_{3,1}(x-t-1,t) = 3x+1+2t$ as a weight function of $\mathcal{P}_{3,1}$.

any vertex v of T'," and if n is the root of T then we take the vertex replacing n as the root of T'. ∎

The plane trees in $\mathcal{P}_{3,1}$ with their weights are listed in Figure 3.

It is worthwhile to point out that there is a simpler variant of Theorems 2.2 and 2.3. A vertex j of a plane tree T is called *really elder* if j has a brother k to its right such that $k < j$. Let $\text{reld}_T(v)$ be the number of really elder children of v. Namely, $\text{reld}_T(v) = \text{gdes}(a_1 \cdots a_m)$, where $a_1, \ldots, a_m$ are all the children of v. Assume $e = (i,j)$ is an edge of a tree T, we say that e is a *really proper* edge, if j is a really elder child of i or $i < \beta_T(j)$. Otherwise, we call e a *really improper* edge. Let

$$\overline{\text{young}}_T(v) = \deg_T(v) - \text{reld}_T(v),$$

and let $\overline{\mathcal{P}}_{n,k}$ (respectively $\overline{\mathcal{O}}_{n,k}$) denote the set of plane trees (respectively plane trees with root 1) on $[n]$ with k really improper edges.

Corollary 2.4. *There holds*

$$Q_{n,k}(x,t) = \sum_{T \in \overline{\mathcal{O}}_{n+1,k}} x^{\overline{\text{young}}_T(1)-1} t^{\text{reld}(T)} = \sum_{T \in \overline{\mathcal{P}}_{n,k}} (x+t+1)^{\overline{\text{young}}_T(1)-1} t^{\text{reld}(T)}.$$

Proof. It suffices to construct a bijection ϕ from $\mathcal{P}_{n,k}$ (respectively $\mathcal{O}_{n+1,k}$) to $\overline{\mathcal{P}}_{n,k}$ (respectively $\overline{\mathcal{O}}_{n+1,k}$). Starting from a plane tree $T \in \mathcal{P}_{n,k}$ (respectively $T \in \mathcal{O}_{n+1,k}$), we define $\phi(T)$ as the plane tree obtained from T as follows. For any vertex v of T with subtrees $T_1, \ldots, T_m$ rooted at $v_1, \ldots, v_m$, respectively, we reorder these subtrees as $T_{\sigma(1)}, \ldots, T_{\sigma(m)}$ such that $v_{\sigma(i)} < v_{\sigma(j)}$ if and only if $\beta_T(v_i) < \beta_T(v_j)$, where $\sigma \in \mathfrak{S}_m$. See Figure 4. ∎

To end this section we make a connection to increasing (plane) trees. A (plane) tree on $[n]$ is called *increasing* if any path from the root to another vertex forms an increasing sequence. Clearly an increasing plane tree has no improper edges and vice versa. Combining (1.6), (1.9) and Theorem 2.2 we get the following result.

Proposition 2.5. *For every $n \geq 1$, we have*

$$\sum_{T \in \mathcal{P}_{n,0}} x^{\text{young}_T(1)} t^{\text{eld}(T)} = \prod_{k=0}^{n-2} (x+k+kt).$$

In particular, the number of increasing trees on $[n]$ is $(n-1)!$ and the number of increasing plane trees on $[n]$ is $(2n-3)!!$.
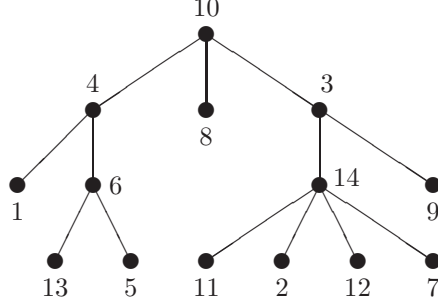
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Figure 4: The plane tree in $\overline{\mathcal{P}}_{14,6}$ corresponding to that in Figure 1.

3   Combinatorial interpretations in half-mobile trees

The notion of half-mobile trees was introduced by Chapoton [4]. A *half-mobile tree* on $[n]$ is defined to be a rooted tree with two kinds vertices, called *labeled* and *unlabeled* (or white and black, respectively) vertices satisfying the following conditions:

- The labeled vertices are in bijection with $[n]$;

- Each unlabeled vertex has at least two children and all of them are labeled;

- There is a fixed cyclic order on the children of any unlabeled vertex;

- The children of each labeled vertex are not ordered.

Let T be a half-mobile tree. For any black vertex x of T, we define $\beta_T(x)$ to be the smallest vertex among all the white descendants of x. From now on, we assume that the rightmost child x of a black vertex v has the smallest $\beta_T(x)$.

Definition 3.1. *An edge $e = (u, v)$ of a half-mobile tree T is called improper if u and v are labeled and $u > \beta_T(v)$, or u is unlabeled and v is its rightmost child, moreover u has a (labeled) father greater than $\beta_T(v)$.*

A *forest of half-mobile trees* on $[n]$ is a graph of which the connected components are half-mobile trees and the white vertex set is $[n]$. Denote by $\mathcal{H}_n$ the set of forests of half-mobile trees on $[n]$. For any $F \in \mathcal{H}_n$, let $\text{imp}(F)$ be the number of improper edges of F, and $\text{tree}(F)$ the number of half-mobile trees of F. Finally define the *black degree* of F, denoted by $\text{bdeg}(F)$, to be the total degree of black vertices minus the number of black vertices. Namely,

$$\text{bdeg}(F) = \sum_{v} (\deg_F(v) - 1),$$

where the sum is over all black vertices v of F.

We first recall a fundamental transformation ψ on $\mathfrak{S}_n$. We identify each permutation $\pi \in \mathfrak{S}_n$ with the sequence $\pi(1)\pi(2)\ldots\pi(n)$. Since a variant of this bijection can be found in [11, Proposition 1.3.1], we only give an informal description of ψ as follows:

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(a) Factorize the permutation π into a product of disjoint cycles.

(b) Order the cycles of π in increasing order of their minima.

(c) Write the minimum at last within each cycle and erase the parentheses of cycles.

For instance, if the factorization of $\pi \in \mathfrak{S}_8$ is $(241)(73)(5)(86)$, then $\psi(\pi) = 2\,4\,1\,7\,3\,5\,8\,6$.

Lemma 3.2. *The mapping ψ is a bijection on $\mathfrak{S}_n$ such that the number of cycles of π is equal to the number of right-to-left minima of $\psi(\pi)$.*

Now we are ready to construct our bijection from the plane trees to forests of half-mobile trees.

Proposition 3.3. *There is a bijection $\theta : \mathcal{O}_{n+1} \longrightarrow \mathcal{H}_n$ such that for any $T \in \mathcal{O}_{n+1}$*

$$\text{young}_T(1) = \text{tree}(\theta(T)), \quad \text{eld}(T) = \text{bdeg}(\theta(T)), \quad \text{imp}(T) = \text{imp}(\theta(T)).$$

Proof. Let $T \in \mathcal{O}_{n+1}$. Pick any vertex v of T with $\deg(v) > 0$. Suppose its children are $a_1, \ldots, a_m$. Consider the permutation $\pi_v = b_1 \cdots b_m$, where $b_j = \beta_T(a_j)$ for $1 \leq j \leq m$. Let $b_{i_1}, b_{i_2}, \ldots, b_{i_r} = b_m$ be all the right-to-left minima of π_v. For any k $(1 \leq k \leq r)$, if $i_k - i_{k-1} > 1$ $(i_0 = 0)$, add a black vertex $\blacksquare$ to be a new child of v and then move the subtrees rooted at $a_{i_{k-1}+1}, \ldots, a_{i_k}$ to be subtrees of this black vertex $\blacksquare$ with the cyclic order $(a_{i_{k-1}+1}, \ldots, a_{i_k})$. In other words, the cycle $(a_{i_{k-1}+1}, \ldots, a_{i_k})$ is a cycle of length greater than 1 in the permutation $\psi^{-1}(\pi_v)$, obtained by applying the inverse mapping of ψ in Lemma 3.2. Since all the elements except the rightmost one in $(a_{i_{k-1}+1}, \ldots, a_{i_k})$ are elder vertices in T, applying the above procedure to every vertex v with $\deg(v) > 0$, we obtain a half-mobile tree T' on $[n+1]$ with root 1. By Lemma 3.2, it is easy to see that the mapping $T \mapsto T'$ is reversible. Furthermore,

$$\text{young}_T(1) = \deg_{T'}(1), \quad \text{eld}(T) = \text{bdeg}(T'), \quad \text{imp}(T) = \text{imp}(T').$$

Deleting the root 1, and shifting the label i to $i - 1$ $(2 \leq i \leq n + 1)$ in T', we obtain a forest $\theta(T)$ of half-mobile trees on $[n]$. This completes the proof. $\blacksquare$

As an illustration of the bijection θ, we apply θ to a tree T in Figure 5 with $n = 14$. For example, the root 1 of T has three children 4, 8, 3 and $\pi_1 = 2\,8\,3$. Note that $\text{young}_T(1) = 2$, $\text{eld}(T) = 5$ and $\text{imp}(T) = 4$.

Let $\mathcal{H}_{n,k}$ denote the set of forests of half-mobile trees on $[n]$ with k improper edges. The following statement follows immediately from Theorem 2.2 and Proposition 3.3.

Theorem 3.4. *For $n \geq 1$, there holds*

$$Q_{n,k}(x, t) = \sum_{F \in \mathcal{H}_{n,k}} x^{\text{tree}(F)-1} t^{\text{bdeg}(F)}.$$

In other words, we have

$$Q_n(x, y, 1, t) = \sum_{F \in \mathcal{H}_n} x^{\text{tree}(F)-1} y^{\text{imp}(F)} t^{\text{bdeg}(F)}.$$
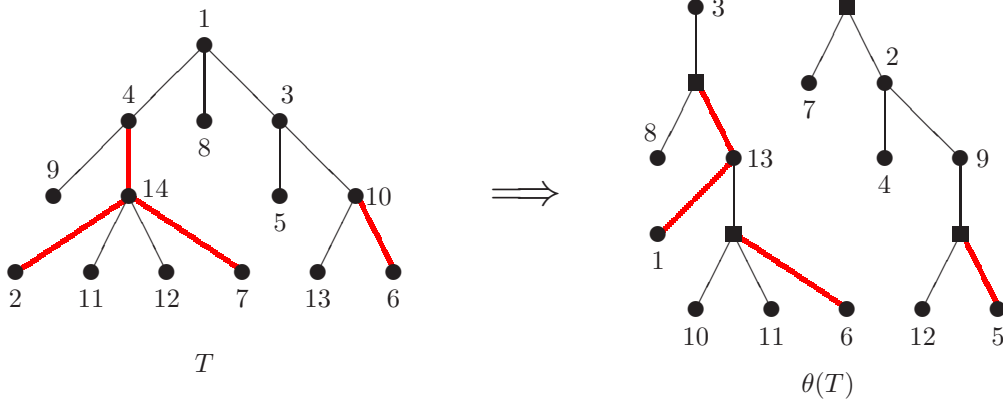
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Figure 5: An example for the bijection θ.

4 Enumeration of plane trees

We first state a variant of Chu-Vandermonde formula.

Lemma 4.1. *For $n \geq 1$, there holds*

$$\sum_{k=0}^{n} \binom{n}{k} \prod_{i=0}^{k} (x+it) \prod_{j=0}^{n-k-1} (y+jt) = x \prod_{k=1}^{n} (x+y+kt). \tag{4.1}$$

Proof. Write the left-hand side of (4.1) as

$$(-1)^n n! x t^n \sum_{k=0}^{n} \binom{-x/t-1}{k} \binom{-y/t}{n-k}$$

and then apply the Chu-Vandermonde convolution formula $\sum_{k=0}^{n} \binom{a}{k} \binom{b}{n-k} = \binom{a+b}{n}$. ∎

Let T be a plane tree containing the edge (i, j). We define a mapping $T \mapsto T'$, called (i, j)-*contraction*, by contracting the edge (i, j) to the vertex i and moving all the children of j to i such that if $a_1, \ldots, a_r$ (respectively $b_1, \ldots, b_s$) are all the children of i to the *left* (respectively *right*) of j and $c_1, \ldots, c_t$ are all the children of j, then the children of i in T' are ordered as $a_1, \ldots, a_r, c_1, \ldots, c_t, b_1, \ldots, b_s$. An illustration of this contraction is given in Figure 6.

Two plane trees T_1 and T_2 are said to be i-*equivalent* and denoted $T_1 \simeq T_2$, if T_2 can be obtained from T_1 by reordering the children of the vertex i (see Figure 7). Moreover, T_1 and T_2 are said to be (i, j)-*equivalent* and denoted $T_1 \sim T_2$, if both T_1 and T_2 contain the edge (i, j) and their images under the (i, j)-contraction are i-equivalent. For instance, the two plane trees T_3 and T_4 in Figure 8 are $(2, 5)$-equivalent, since after the $(2, 5)$-contraction they become respectively T_1 and T_2 in Figure 7.
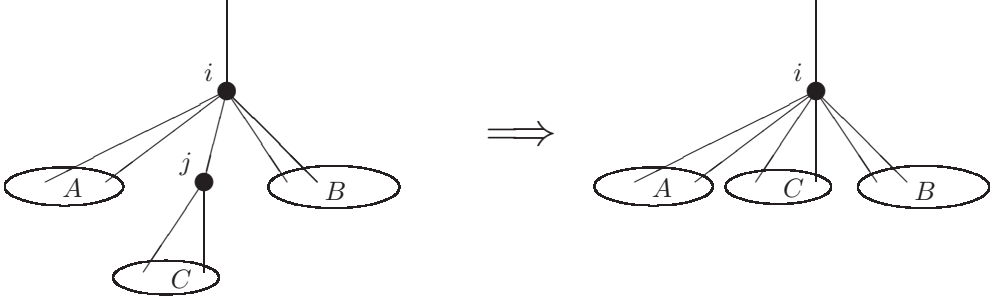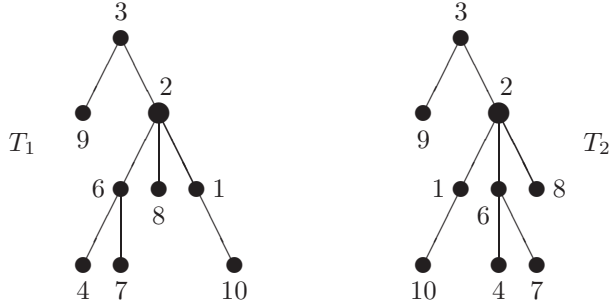
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Figure 6: The (i,j)-contraction.

Figure 7: Two 2-equivalent plane trees.

Lemma 4.2. *Let T_0 be a plane tree on $[n]$ containing the edge (i,j). Let $\mathcal{T}_0(i,j)$ be the set of plane trees obtained from the (i,j)-contraction of those i-equivalent to T_0. Then*

$$\sum_{T \sim T_0} t^{\mathrm{eld}(T)} \prod_{k=1}^{n} x_k^{\mathrm{young}_T(k)} = x_i \sum_{T \in \mathcal{T}_0(i,j)} (x_i + x_j + t)^{\mathrm{young}_T(i)} t^{\mathrm{eld}(T)} \prod_{k \in [n] \setminus \{i,j\}} x_k^{\mathrm{young}_T(k)}.$$

Proof. Suppose that $\deg_{T_0}(i) + \deg_{T_0}(j) = m + 1$. By Eq. (2.1) and Lemma 4.1, we have

$$\begin{aligned}
\sum_{T \simeq T_0} t^{\mathrm{eld}_T(i)} x_i^{\mathrm{young}_T(i)} t^{\mathrm{eld}_T(j)} x_j^{\mathrm{young}_T(j)} &= \sum_{k=0}^{m} \sum_{\substack{T \sim T_0 \\ \deg_T(i) = k+1}} t^{\mathrm{eld}_T(i)} x_i^{\mathrm{young}_T(i)} t^{\mathrm{eld}_T(j)} x_j^{\mathrm{young}_T(j)} \\
&= \sum_{k=0}^{m} \binom{m}{k} \prod_{r=0}^{k} (x_i + rt) \cdot \prod_{s=0}^{m-k-1} (x_j + st) \\
&= x_i \prod_{k=1}^{m} (x_i + x_j + kt) \\
&= x_i \sum_{T \in \mathcal{T}_0(i,j)} (x_i + x_j + t)^{\mathrm{young}_T(i)} t^{\mathrm{eld}_T(i)}.
\end{aligned}$$

Multiplying by $\prod_{k \in [n] \setminus \{i,j\}} t^{\mathrm{eld}_{T_0}(k)} x_k^{\mathrm{young}_{T_0}(k)}$ we get the desired result. $\blacksquare$
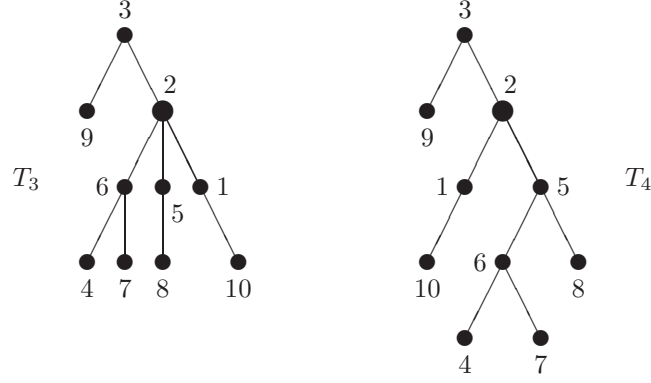
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Figure 8: Two $(2,5)$-equivalent plane trees in $\mathcal{P}_{10}$.

Let $\mathcal{P}_n$ denote the set of all plane trees on $[n]$.

Theorem 4.3. *For $n \geq 1$, there holds*

$$\sum_{T \in \mathcal{P}_n} t^{\mathrm{eld}(T)} \prod_{i=1}^{n} x_i^{\mathrm{young}_T(i)} = \prod_{k=0}^{n-2} (x_1 + \cdots + x_n + kt). \tag{4.2}$$

Proof. We shall prove (4.2) by induction on n. The identity obviously holds for $n = 1$. Suppose (4.2) holds for $n - 1$. For $i, j \in [n]$ ($i \neq j$), let $\mathcal{P}_n(i,j)$ be the set of all plane trees on $[n]$ containing the edge (i,j). Let $\mathcal{A}$ be a maximal set of plane trees in $\mathcal{P}_n(i,j)$ that are pairwise not (i,j)-equivalent and $\mathcal{B}$ the set of plane trees on $[n] \setminus \{j\}$ obtained from the (i,j)-contraction of those in $\mathcal{A}$.

By Lemma 4.2, we have

$$\sum_{T \in \mathcal{P}_n(i,j)} t^{\mathrm{eld}(T)} \prod_{k=1}^{n} x_k^{\mathrm{young}_T(k)} = \sum_{T_0 \in \mathcal{A}} \sum_{T \sim T_0} t^{\mathrm{eld}(T)} \prod_{k=1}^{n} x_k^{\mathrm{young}_T(k)}$$

$$= x_i \sum_{T_0' \in \mathcal{B}} \sum_{T \simeq T_0'} (x_i + x_j + t)^{\mathrm{young}_T(i)} t^{\mathrm{eld}(T)} \prod_{k \in [n] \setminus \{i,j\}} x_k^{\mathrm{young}_T(k)}.$$

By the induction hypothesis, the last double sum is equal to

$$x_i \sum_{T \in \mathcal{P}_{[n] \setminus \{j\}}} (x_i + x_j + t)^{\mathrm{young}_T(i)} t^{\mathrm{eld}(T)} \prod_{k \in [n] \setminus \{i,j\}} x_k^{\mathrm{young}_T(k)}$$

$$= x_i \prod_{k=0}^{n-3} (x_1 + \cdots + x_n + (k+1)t).$$

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Summing over all of the pairs (i,j) with $i \neq j$ we get

$$\sum_{\substack{1 \leq i,j \leq n \\ i \neq j}} \sum_{T \in \mathcal{P}_n(i,j)} t^{\mathrm{eld}(T)} \prod_{k=1}^{n} x_k^{\mathrm{young}_T(k)}$$

$$= (n-1)(x_1 + \cdots + x_n) \prod_{k=0}^{n-3} (x_1 + \cdots + x_n + (k+1)t)$$

$$= (n-1) \prod_{k=0}^{n-2} (x_1 + \cdots + x_n + kt). \tag{4.3}$$

Noticing that any plane tree on $[n]$ has $n-1$ edges, we have

$$\sum_{\substack{1 \leq i,j \leq n \\ i \neq j}} \sum_{T \in \mathcal{P}_n(i,j)} t^{\mathrm{eld}(T)} \prod_{k=1}^{n} x_k^{\mathrm{young}_T(k)} = (n-1) \sum_{T \in \mathcal{P}_n} t^{\mathrm{eld}(T)} \prod_{k=1}^{n} x_k^{\mathrm{young}_T(k)}. \tag{4.4}$$

Comparing (4.3) and (4.4) yields the desired formula for $\mathcal{P}_n$. ∎

Corollary 4.4. *The number of unlabeled plane trees on $n+1$ vertices is the Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$.*

Proof. Setting $x_1 = \cdots = x_n = t = 1$ in (4.2) gives the number of labeled plane trees on n vertices, i.e., $|\mathcal{P}_n| = (2n-2)!/(n-1)!$. We then obtain the number of unlabeled plane trees on n vertices by dividing $|\mathcal{P}_n|$ by $n!$. ∎

Corollary 4.5. *The number of unlabeled plane trees with k leaves on $n+1$ vertices is the Narayana number $N_{n,k} = \frac{1}{n}\binom{n}{k}\binom{n}{k-1}$.*

Proof. Setting $t = 1$ in (4.2) and replacing n by $n+1$, we have

$$\sum_{T \in \mathcal{P}_{n+1}} \prod_{i=1}^{n+1} x_i^{\mathrm{young}_T(i)} = \prod_{s=0}^{n-1} (x_1 + \cdots + x_{n+1} + s).$$

Note that the vertex i is a leaf of the plane tree T if and only if $\mathrm{young}_T(i) = 0$. Hence the number $a_{n,k}$ of plane trees on $[n+1]$ with leaves $1, 2, \ldots, k$ is the sum of the coefficients of monomials $x_{k+1}^{r_{k+1}} \cdots x_{n+1}^{r_{n+1}}$ with $r_j \geq 1$ for all $k+1 \leq j \leq n+1$ in the expansion of the symmetric polynomial

$$P = \prod_{s=0}^{n-1} (x_{k+1} + \cdots + x_{n+1} + s).$$

Note that the sum of the coefficients of monomials in P not containing i ($0 \leq i \leq n+1-k$) fixed variables is $\prod_{s=0}^{n-1}(n+1-k-i+s)$. By the Principle of Inclusion-Exclusion and the

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Chu-Vandermonde convolution formula, one sees that

$$
\begin{aligned}
a_{n,k} &= \sum_{i=0}^{n-k+1} (-1)^i \binom{n-k+1}{i} \prod_{s=0}^{n-1} (n+1-k-i+s) \\
&= n! \sum_{i=0}^{n-k+1} (-1)^i \binom{n-k+1}{i} \binom{2n-k-i}{n} \\
&= n! \binom{n-1}{k-1}.
\end{aligned}
$$

Therefore, the number of unlabeled plane trees with k leaves on $n+1$ vertices is equal to

$$
\frac{a_{n,k}}{k!(n-k+1)!} = \frac{n!}{k!(n-k+1)!} \binom{n-1}{k-1} = N_{n,k}. \qquad \blacksquare
$$

Let $\mathcal{P}_n^{(r)}$ denote the set of plane trees on $[n]$ with a specific root $r \in [n]$. Then we can refine Theorem 4.3 as follows.

Theorem 4.6. *For $n \geq 1$, there holds*

$$
\sum_{T \in \mathcal{P}_n^{(r)}} t^{\mathrm{eld}(T)} \prod_{i=1}^{n} x_i^{\mathrm{young}_T(i)} = x_r \prod_{k=1}^{n-2} (x_1 + \cdots + x_n + kt). \tag{4.5}
$$

Proof. Suppose Eq. (4.5) hold for $n-1$. Let $\mathcal{P}_n^{(r)}(i,j)$ denote the subset of plane trees in $\mathcal{P}_n(i,j)$ with root r. Similarly to the proof of Theorem 4.3, we can show that

$$
\sum_{T \in \mathcal{P}_n^{(r)}(i,j)} t^{\mathrm{eld}(T)} \prod_{i=1}^{n} x_i^{\mathrm{young}_T(i)} =
\begin{cases}
x_r x_i \prod_{k=1}^{n-3} (x_1 + \cdots + x_n + (k+1)t), & \text{if } i \neq r, \\
(x_r + x_j + t) x_i \prod_{k=1}^{n-3} (x_1 + \cdots + x_n + (k+1)t), & \text{if } i = r.
\end{cases}
$$

The proof then follows from computing the following sum:

$$
\sum_{\substack{1 \leq i,j \leq n \\ i \neq j}} \sum_{T \in \mathcal{P}_n^{(r)}(i,j)} t^{\mathrm{eld}(T)} \prod_{i=1}^{n} x_i^{\mathrm{young}_T(i)},
$$

and using the fact that any plane trees on $[n]$ has $n-1$ edges. $\qquad \blacksquare$

Corollary 4.7. *For $1 \leq r < s \leq n$, there holds*

$$
x_r^{-1} \sum_{T \in \mathcal{P}_n^{(r)}} t^{\mathrm{eld}(T)} \prod_{i=1}^{n} x_i^{\mathrm{young}_T(i)} = x_s^{-1} \sum_{T \in \mathcal{P}_n^{(s)}} t^{\mathrm{eld}(T)} \prod_{i=1}^{n} x_i^{\mathrm{young}_T(i)}.
$$

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It would be interesting to have a combinatorial proof of the above identity. We give such a proof in the case $(r, s) = (1, 2)$. Let T be any plane tree in $\mathcal{P}_n^{(1)}$. Suppose all the children of the root 1 are $a_1, \ldots, a_m$ and 2 is a descendant of a_t. Hence, $a_1, \ldots, a_{t-1}$ are elder, while a_t is younger. Assume all the children of 2 are $b_1, \ldots, b_l$. Exchanging the subtrees with roots $a_{t+1}, \ldots, a_m$ in T and the subtrees with roots $b_1, \ldots, b_l$ in their previous orders, and then exchanging the labels 1 and 2, we obtain a plane tree $T' \in \mathcal{P}_n^{(2)}$. It is easy to see that the mapping $T \mapsto T'$ is a bijection from $\mathcal{P}_n^{(1)}$ to $\mathcal{P}_n^{(2)}$. Moreover, one sees that $\mathrm{eld}(T) = \mathrm{eld}(T')$, $\mathrm{young}_T(1) = \mathrm{young}_{T'}(1) + 1$, $\mathrm{young}_T(2) = \mathrm{young}_{T'}(2) - 1$, and $\mathrm{young}_T(i) = \mathrm{young}_{T'}(i)$ if $i \neq 1, 2$.

As applications of Theorem 4.6 we derive three classical results on planted and plane forests, which correspond, respectively, to Theorem 5.3.4, Corollary 5.3.5 and Theorem 5.3.10 in Stanley's book [12].

Recall that a *planted forest* σ (or, *rooted forest*) is a graph whose connected components are rooted trees. If the vertex set of σ is $[n]$, then we define the *ordered degree sequence* $\delta(\sigma) = (d_1, \ldots, d_n)$, where $d_i = \deg(i)$.

Corollary 4.8. *Let* $\mathbf{d} = (d_1, \ldots, d_n) \in \mathbb{N}^n$ *with* $\sum d_i = n - k$. *Then the number of planted forests on* $[n]$ *(necessarily with* k *components) with ordered degree sequence* $\mathbf{d}$ *is given by*

$$\binom{n-1}{k-1}\binom{n-k}{d_1, \ldots, d_n}.$$

Proof. Equating the coefficients of $x_1^k x_2^{d_1} \cdots x_{n+1}^{d_n}$ in (4.5) with $r = 1$, $t = 0$ and n replaced by $n + 1$, we get the desired result. ∎

Given a planted forest σ, define the *type* of σ to be the sequence

$$\mathrm{type}\, \sigma = (r_0, r_1, \ldots),$$

where r_i vertices of σ have degree i. Similarly we can define the *type* for a plane forest. As is well-known (see [12, p. 30]), Corollary 4.8 can be restated in the following equivalent form.

Corollary 4.9. *Let* $\mathbf{r} = (r_0, \ldots, r_m) \in \mathbb{N}^{m+1}$ *with* $\sum r_i = n$ *and* $\sum(1 - i)r_i = k > 0$. *Then the number of planted forests on* $[n]$ *(necessarily with* k *components) of type* $\mathbf{r}$ *is given by*

$$\binom{n-1}{k-1}\frac{(n-k)!}{0!^{r_0} \cdots m!^{r_m}}\binom{n}{r_0, \ldots, r_m}.$$

A *plane forest* is a family of plane trees in which the roots of the plane trees are linearly ordered.

Corollary 4.10. *Let* $\mathbf{r} = (r_0, \ldots, r_m) \in \mathbb{N}^{m+1}$ *with* $\sum r_i = n$ *and* $\sum(1 - i)r_i = k > 0$. *Then the number of unlabeled plane forests on* $[n]$ *(necessarily with* k *components) of type* $\mathbf{r}$ *is given by*

$$\frac{k}{n}\binom{n}{r_0, \ldots, r_m}.$$

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Proof. Let us see how many plane forests we can obtain from a planted forest σ of type **r** by ordering the vertices. For any vertex v of σ, there are $\deg(v)!$ ways to linearly order its children. By definition, there are r_i vertices of σ having degree i. Besides, there are $k!$ ways to linearly order the trees of σ. So, we can obtain $0!^{r_0} \cdots m!^{r_m} k!$ plane forests from σ. Hence it follows from Corollary 4.9 that the total number of plane forests on $[n]$ of type **r** is equal to

$$\binom{n-1}{k-1} \frac{(n-k)!}{0!^{r_0} \cdots m!^{r_m}} \binom{n}{r_0, \ldots, r_m} 0!^{r_0} \cdots m!^{r_m} k! = (n-1)! k \binom{n}{r_0, \ldots, r_m}. \qquad (4.6)$$

On the other hand, every unlabeled plane forest with n vertices has $n!$ different labeling methods. Dividing (4.6) by $n!$ yields the desired formula. ∎

5 Forests of plane trees

When $x = r$ is an integer, we can give another interpretation for the polynomial $rQ_{n-r,k}(r,t)$ in terms of forests. Let $\mathcal{F}^r_{n,k}$ denote the set of forests of r plane trees on $[n]$ with k improper edges and with roots $1, \ldots, r$. The following is a generalization of a theorem of Shor [10], which corresponds to the case $t = 0$.

Theorem 5.1. *The generating function for forests of r plane trees on $[n]$ with k improper edges and with roots $1, \ldots, r$ by number of elder vertices is $rQ_{n-r,k}(r,t)$. Namely,*

$$rQ_{n-r,k}(r,t) = \sum_{F \in \mathcal{F}^r_{n,k}} t^{\mathrm{eld}(F)}. \qquad (5.1)$$

Proof. We proceed by induction on n. Identity (5.1) obviously holds for $n = r+1$. Suppose (5.1) holds for n. Similarly to the proof of 2.2, we can show that

$$\sum_{T \in \mathcal{F}^r_{n+1,k}[\deg(n+1)=0]} t^{\mathrm{eld}(T)} = [n + t(n-r)] \sum_{F \in \mathcal{F}^r_{n,k}} t^{\mathrm{eld}(T)}, \qquad (5.2)$$

$$\sum_{F \in \mathcal{F}_{n+1,k}[\deg(n+1)>0]} t^{\mathrm{eld}(T)} = kt \sum_{F \in \mathcal{F}^r_{n,k}} t^{\mathrm{eld}(T)} + (n+k-r-1) \sum_{F \in \mathcal{P}^r_{n,k-1}} t^{\mathrm{eld}(T)}. \qquad (5.3)$$

Summing (5.2) and (5.3) and using the induction hypothesis and (1.10), we obtain

$$\sum_{T \in \mathcal{F}^r_{n+1,k}} t^{\mathrm{eld}(T)} = [n + t(n+k-r)] rQ_{n-r,k}(r,t) + (n+k-r-1) rQ_{n-r,k-1}(r,t),$$

$$= rQ_{n+1-r,k}(r,t).$$

Namely, Eq. (5.1) holds for $n+1$. This completes the proof. ∎

6 Proof of the duality formula for Q_n

First we restate the duality formula (1.7) in terms of $Q_{n,k}(x,t)$.

Lemma 6.1. *For $n \geq 2$, the duality formula* (1.7) *is equivalent to*

$$Q_{n,k}(x,t) = (x - k + t + 1)Q_{n-1,k}(x + t + 1, t) + (n + k - 2)Q_{n-1,k-1}(x + t + 1, t). \quad (6.1)$$

Proof. Plugging (1.9) into (1.7) we see that (1.7) is equivalent to the following recurrence relation for $Q_{n,k}(x,t)$:

$$Q_{n,k}(x,t) = Q_{n,k}(-(x + n + nt)/t, 1/t)(-t)^{n-k-1}. \quad (6.2)$$

Setting $X_n = -(x + n + nt)/t$ and $T = 1/t$, by means of (1.10) we have

$$Q_{n,k}(X_n, T) = -\frac{x - k + t + 1}{t} Q_{n-1,k}(X_n, T) + (n + k - 2)Q_{n-1,k-1}(X_n, T). \quad (6.3)$$

Multiplying (6.3) by $(-t)^{n-k-1}$, we see that (6.2) is equivalent to (6.1). ∎

Now, by replacing x with $x - t - 1$ in (6.1), we get

$$Q_{n,k}(x - t - 1, t) = (x - k)Q_{n-1,k}(x,t) + (n + k - 2)Q_{n-1,k-1}(x,t). \quad (6.4)$$

Subtracting (6.4) from (1.10), we are led to the following equivalent identity:

Lemma 6.2. *For $n \geq 2$ and $0 \leq k \leq n - 1$ there holds*

$$Q_{n,k}(x,t) - Q_{n,k}(x - t - 1, t) = (t + 1)(n + k - 1)Q_{n-1,k}(x,t). \quad (6.5)$$

Proof. We proceed by induction on n. Eq. (6.5) is obviously true for $n = 2$. Suppose it is true for $n - 1$. By the definition (1.10) of $Q_{n,k}(x,t)$ we have

$$\begin{aligned}
Q_{n,k}&(x,t) - Q_{n,k}(x - t - 1, t) \\
&= [x + n - 1 + t(n + k - 2)]Q_{n-1,k}(x,t) + (n + k - 2)Q_{n-1,k-1}(x,t) \\
&\quad - [x + n - 2 + t(n + k - 3)]Q_{n-1,k}(x - t - 1, t) - (n + k - 2)Q_{n-1,k-1}(x - t - 1, t) \\
&= [x + n - 2 + t(n + k - 3)][Q_{n-1,k}(x,t) - Q_{n-1,k}(x - t - 1, t)] \\
&\quad + (t + 1)Q_{n-1,k}(x,t) + (n + k - 2)[Q_{n-1,k-1}(x,t) - Q_{n-1,k-1}(x - t - 1, t)],
\end{aligned}$$

and the induction hypothesis implies that the above quantity is equal to

$$\begin{aligned}
(t + 1)&[x + n - 2 + t(n + k - 3)](n + k - 2)Q_{n-2,k}(x,t) + (t + 1)Q_{n-1,k}(x,t) \\
&\quad + (t + 1)(n + k - 2)(n + k - 3)Q_{n-2,k-1}(x,t) \\
&= (t + 1)(n + k - 2)\{[x + n - 2 + t(n + k - 3)]Q_{n-2,k}(x,t) + (n + k - 3)Q_{n-2,k-1}(x,t)\} \\
&\quad + (t + 1)Q_{n-1,k}(x,t) \\
&= (t + 1)(n + k - 2)Q_{n-1,k}(x,t) + (t + 1)Q_{n-1,k}(x,t) \\
&= (t + 1)(n + k - 1)Q_{n-1,k}(x,t).
\end{aligned}$$

Thus (6.5) is true for n. This completes the proof. ∎

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Remark. We can also prove Theorem 1.1 by arguing directly with Q_n instead of $Q_{n,k}$. Indeed, the theorem is equivalent to saying that

$$F_{n+1}(x - z - t) = [x + nz + (y - z)(n + y\partial_y)]F_n(x), \qquad (6.6)$$

where $F_n(x) = Q_n(x, y, z, t)$. Subtracting (6.6) from the definition (1.4) yields

$$F_{n+1}(x) - F_{n+1}(x - z - t) = (z + t)(n + y\partial_y)F_n(x). \qquad (6.7)$$

By (1.4) and the induction hypothesis, the left-hand side of (6.7) may be written as

$$\begin{aligned}
&[x + nz + (y + t)(n + y\partial_y)]F_n(x) - [x - z - t + nz + (y + t)(n + y\partial_y)]F_n(x - z - t) \\
&= [x + (n - 1)z - t][F_n(x) - F_n(x - z - t)] + (z + t)F_n(x) \\
&\quad + (y + t)(n + y\partial_y)[F_n(x) - F_n(x - z - t)] \\
&= (z + t)\left\{[x + (n - 1)z - t](n - 1 + y\partial_y) + (y + t)(n + y\partial_y)(n - 1 + y\partial_y)\right\}F_{n-1}(x) \\
&\quad + (z + t)F_n(x).
\end{aligned}$$

Applying the differential operator identity

$$(y + t)(n + y\partial_y)(n - 1 + y\partial_y) = (n - 1 + y\partial_y)[(y + t)(n + y\partial_y) - y],$$

to the last expression, we obtain

$$\begin{aligned}
&(z + t)(n - 1 + y\partial_y)\left\{[x + (n - 1)z + (y + t)(n - 1 + y\partial_y)]F_{n-1}(x)\right\} + (z + t)F_n(x) \\
&= (z + t)(n + y\partial_y)F_n(x),
\end{aligned}$$

as desired.

7   Open problems

Although we have shown the fecundity of polynomials Q_n in the enumeration of plane trees and forests, there are still further interesting problems.

First of all, is there any connection between these polynomials and the Lambert W function or a generalization of the Lambert W function? In particular, is there an analogue of the formula (1.1) or (1.3) for Q_n?

Secondly, it would be interesting to have a better combinatorial understanding of the polynomials Q_n. For example, from Theorems 2.2 and 2.3 we deduce that

$$\sum_{T \in \mathcal{O}_{n+1,k}} x^{\mathrm{young}_T(1)-1} t^{\mathrm{eld}(T)} = \sum_{T \in \mathcal{P}_{n,k}} (x + t + 1)^{\mathrm{young}_T(1)} t^{\mathrm{eld}(T)}. \qquad (7.1)$$

Is there a direct combinatorial proof of (7.1)? Also, the duality formula (1.7) deserves a combinatorial proof. For $t = 0$, such proofs have been given by Chen and Guo [6].

Thirdly, since our proof of Theorems 4.3 and 4.6 is by induction, it remains to find direct bijective proofs of them.

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Finally, using a dual statistic, the number of "improper vertices," Gessel and Seo [8] have recently given several combinatorial interpretations of the polynomials

$$xQ_n(x, z, z, t - z) = x \prod_{k=1}^{n-1} (x + (n - k)z + kt). \tag{7.2}$$

On the other hand, we derive from Eqs. (1.8), (1.9) and Theorem 2.2 that the same polynomials (7.2) also have the following expression:

$$\sum_{T \in \mathcal{P}_{n+1}^{(1)}} x^{\mathrm{young}_T(1)} (t - z)^{\mathrm{eld}(T)} z^{n - \mathrm{young}_T(1) - \mathrm{eld}(T)}.$$

It might be of interest to have a combinatorial explanation for these different interpretations.

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